

5.1

Polygon Sum Conjecture

In this lesson you will

- Discover a formula for finding the **sum of the angle measures** for any **polygon**
- Use **deductive reasoning** to explain why the polygon sum formula works

Triangles come in many different shapes and sizes. However, as you discovered in Chapter 4, the sum of the angle measures of any triangle is 180° . In this lesson you will investigate the sum of the angle measures of other polygons. After you find a pattern, you'll write a formula that relates the number of sides of a polygon to the sum of the measures of its angles.



Investigation: Is There a Polygon Sum Formula?

Draw three different quadrilaterals. For each quadrilateral, carefully measure the four angles and then find the sum of the angle measures. If you measure carefully, you should find that all of your quadrilaterals have the same angle sum. What is the sum? Record it in a table like the one below.

Number of sides	3	4	5	6	7	8	...	n
Sum of angle measures	180°				900°	1080°		

Next draw three different pentagons. Carefully measure the angles in each pentagon and find the angle sum. Again, you should find that the angle sum is the same for each pentagon. Record the sum in the table.

Use your findings to complete the conjectures below.

Quadrilateral Sum Conjecture The sum of the measures of the four angles of any quadrilateral is _____ **C-29**

Pentagon Sum Conjecture The sum of the measures of the five angles of any pentagon is _____ **C-30**

Now draw at least two different hexagons and find their angle sum. Record the sum in the table.

The angle sums for heptagons and octagons have been entered in the table for you, but you can check the sums by drawing and measuring your own polygons.

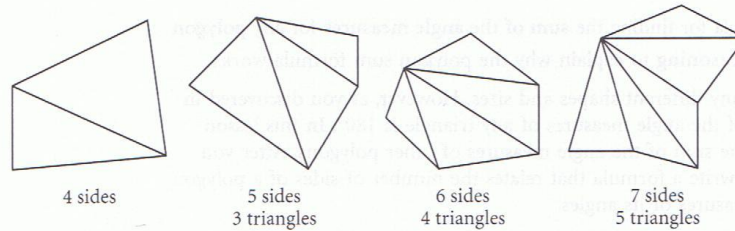
Look for a pattern in the completed table. Find a general formula for the sum of the angle measures of a polygon in terms of the number of sides, n . (*Hint:* Use what you learned in Chapter 2 about finding the formula for a pattern with a constant difference.) Then complete this conjecture.

Polygon Sum Conjecture The sum of the measures of the n interior angles of an n -gon is _____ **C-31**

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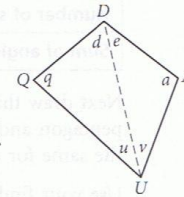
Lesson 5.1 • Polygon Sum Conjecture (continued)

You can use deductive reasoning to see why your formula works. In each polygon below, all the diagonals from one vertex have been drawn, creating triangles. Notice that in each polygon, the number of triangles is 2 less than the number of sides.



The quadrilateral has been divided into two triangles, each with an angle sum of 180° . So, the angle sum for the quadrilateral is $180^\circ \cdot 2$, or 360° . The pentagon has been divided into three triangles, so its angle sum is $180^\circ \cdot 3$, or 540° . The angle sums for the hexagon and the heptagon are $180^\circ \cdot 4$ and $180^\circ \cdot 5$, respectively. In general, if a polygon has n sides, its angle sum is $180^\circ(n - 2)$ or, equivalently, $180^\circ n - 360^\circ$. This should agree with the formula you found earlier.

You can use the diagram at right to write a paragraph proof of the Quadrilateral Sum Conjecture. See if you can fill in the steps in the proof below.

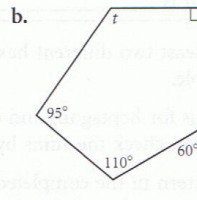
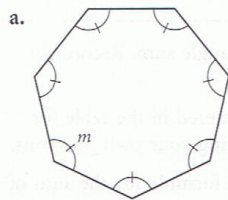


Paragraph Proof: Show that $m\angle Q + m\angle U + m\angle A + m\angle D = 360^\circ$.

$q + d + u = 180^\circ$ and $e + a + v = 180^\circ$ by the _____ Conjecture.
By the Addition Property of Equality, $q + d + u + e + a + v =$ _____.
Therefore, the sum of the angle measures of a quadrilateral is 360° .

Here is an example using your new conjectures.

EXAMPLE Find the lettered angle measures.



- **Solution**
- The polygon has seven sides, so the angle sum is $180^\circ \cdot 5$, or 900° . Because all the angles have the same measure, the measure of angle m is $900^\circ \div 7$, or about 128.6° .
 - The polygon has five sides, so the angle sum is $180^\circ \cdot 3$, or 540° . Therefore, $90^\circ + 120^\circ + 110^\circ + 95^\circ + t = 540^\circ$. Solving for t gives $t = 125^\circ$.

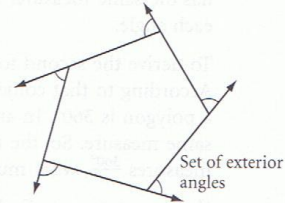
Exterior Angles of a Polygon

In this lesson you will

- Find the sum of the measures of one set of **exterior angles** of a polygon
- Derive two formulas for the measure of each angle of an equiangular polygon

In Lesson 5.1, you discovered a formula for the sum of the measures of the *interior* angles of any polygon. In this lesson, you will find a formula for the sum of the measures of a set of *exterior* angles.

To create a set of exterior angles of a polygon, extend each side of the polygon to form one exterior angle at each vertex.

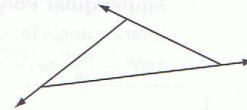


Investigation: Is There an Exterior Angle Sum?

Draw a triangle and extend its sides to form a set of exterior angles.

Measure two of the *interior* angles of your triangle. Then use the Triangle Sum Conjecture to find the measure of the remaining angle. Label each angle in your triangle with its measure.

Use the Linear Pair Conjecture to calculate the measure of each exterior angle. Then find the sum of the exterior angle measures. Record your result in a table like the one below. Do you think you would get the same result for a different triangle? Draw another triangle and see.



Number of sides	3	4	5	6	7	8	...	n
Sum of exterior angle measures	360°						...	

Next, draw a quadrilateral and create a set of exterior angles. Use a procedure similar to the one you used for triangles to find the sum of the set of exterior angles. Add your result to the table.

Next, find the sum of the exterior angles for a pentagon. Are you starting to see a pattern? Predict the sum of the exterior angle measures for a hexagon. Then draw a hexagon and check your prediction. Add your results to the table.

You should have discovered that, no matter how many sides a polygon has, the sum of its exterior angle measures is 360° . This can be stated as a conjecture.

Exterior Angle Sum Conjecture For any polygon, the sum of the measures of a set of exterior angles is 360° . **C-32**

Look at the software construction on page 263 of your book. Notice that the exterior angles stay the same as the polygon shrinks to a point. How does this demonstrate the Exterior Angle Sum Conjecture?

(continued)

Lesson 5.2 • Exterior Angles of a Polygon (continued)

Now you will find two formulas for the measure of each *interior* angle in an equiangular polygon with n sides. Remember, an equiangular polygon is a polygon in which all angles have the same measure.

You can use the Polygon Sum Conjecture to derive the first formula. That conjecture states that the sum of the interior angle measures in a polygon with n sides is $180^\circ(n - 2)$. If the polygon is equiangular, then each of the n angles has the same measure. Use these facts to write a formula for the measure of each angle.

To derive the second formula, you can use the Exterior Angle Sum Conjecture. According to that conjecture, the sum of the measures of the n exterior angles of a polygon is 360° . In an equiangular polygon, each of the exterior angles has the same measure. So, the measure of each exterior angle is $\frac{360^\circ}{n}$. If each exterior angle measures $\frac{360^\circ}{n}$, what must be the measure of each interior angle?

You can state your findings in a conjecture.

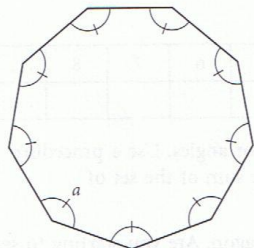
Equiangular Polygon Conjecture You can find the measure of each interior angle in an equiangular n -gon by using either of these formulas:

$$180^\circ - \frac{360^\circ}{n} \text{ or } \frac{180^\circ(n - 2)}{n}$$

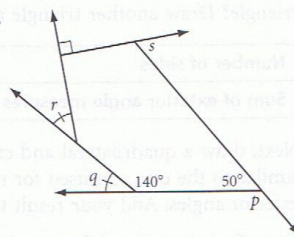
C-33

EXAMPLE Find the lettered angle measures.

a.



b.



► **Solution** a. This is an equiangular 9-gon. You can use the Equiangular Polygon Conjecture to find the measure of a .

$$a = 180^\circ - \frac{360^\circ}{9} = 180^\circ - 40^\circ = 140^\circ$$

b. By the Linear Pair Conjecture, $p = 130^\circ$ and $q = 40^\circ$ so $r = 40^\circ$ as well. To find s , use the Exterior Angle Sum Conjecture.

$$130^\circ + 40^\circ + 40^\circ + 90^\circ + s = 360^\circ$$

$$\text{Solving this equation gives } s = 60^\circ.$$

Kite and Trapezoid Properties

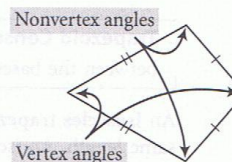
In this lesson you will

- Investigate the properties of **kites**
- Investigate properties of **trapezoids** and **isosceles trapezoids**

In this lesson you will look at two special types of quadrilaterals, kites and trapezoids. Recall that a **kite** is a quadrilateral with two distinct pairs of congruent consecutive sides.

You can make a kite by constructing two different isosceles triangles on opposite sides of a common base and then removing the base. In an isosceles triangle, the angle between the two congruent sides is called the vertex angle. For this reason, we'll call angles between the pairs of congruent sides of a kite **vertex angles**. We'll refer to the other two angles as **nonvertex angles**.

A kite has one line of reflectional symmetry, just like an isosceles triangle. You can use this property to discover other properties of kites.



Investigation 1: What Are Some Properties of Kites?

Follow Step 1 in your book to construct a kite on patty paper.

Compare each angle to the opposite angle by folding. Which angles are congruent, vertex angles or nonvertex angles? Use your findings to complete this conjecture.

Kite Angles Conjecture The _____ angles of a kite are congruent. C-34

Draw the diagonals of the kite. Fold the kite along one of the diagonals. The two parts of the other diagonal should coincide. What can you conclude about the angle between the diagonals? You are now ready to complete this conjecture.

Kite Diagonals Conjecture The diagonals of a kite are _____. C-35

Now fold along each diagonal and compare the lengths of the segments on the diagonals. Does either diagonal bisect the other? Complete this conjecture.

Kite Diagonal Bisector Conjecture The diagonal connecting the vertex angles of a kite is the _____ of the other diagonal. C-36

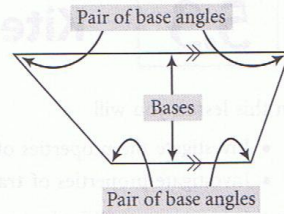
Fold along the diagonal connecting the vertex angles. Does the diagonal bisect the vertex angles? Now fold along the other diagonal. Does it bisect the nonvertex angles? Complete this conjecture.

Kite Angle Bisector Conjecture The _____ angles of a kite are bisected by a diagonal. C-37

(continued)

Lesson 5.3 • Kite and Trapezoid Properties (continued)

Now you will explore some properties of trapezoids. Recall that a **trapezoid** is a quadrilateral with exactly one pair of parallel sides. The parallel sides are called **bases**. A pair of angles that share a base as a common side are called **base angles**.



Investigation 2: What Are Some Properties of Trapezoids?

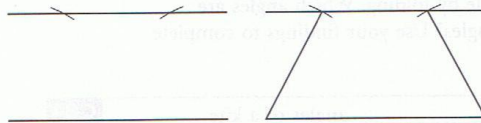
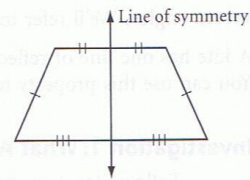
Follow Steps 1 and 2 in your book. Use your findings to complete this conjecture.

Trapezoid Consecutive Angles Conjecture The consecutive angles between the bases of a trapezoid are _____.

C-38

An **isosceles trapezoid** is a trapezoid whose nonparallel sides are the same length. An isosceles trapezoid has a line of symmetry that passes through the midpoints of the two bases.

Use both edges of your straightedge to draw parallel segments. To construct the congruent sides, make identical arcs centered at the endpoints of one of the segments so that each arc intersects the other segment. Then connect points as shown below to form the trapezoid.



Measure each pair of base angles. How do the angles in each pair compare? Complete this conjecture.

Isosceles Trapezoid Conjecture The base angles of an isosceles trapezoid are _____.

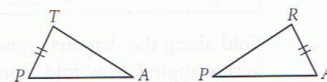
C-39

Now draw the two diagonals. Compare their lengths and complete this conjecture.

Isosceles Trapezoid Diagonals Conjecture The diagonals of an isosceles trapezoid are _____.

C-40

Follow the Developing Proof instructions on page 271 of your book and complete a flowchart proof of the Isosceles Trapezoid Diagonals Conjecture, using the Isosceles Trapezoid Conjecture and congruent triangles. Separating the triangles as shown at right might help you. Remember to mark congruent parts on your diagram.



5.4

Properties of Midsegments

In this lesson you will

- Discover properties of the **midsegment** of a triangle
- Discover properties of the **midsegment** of a trapezoid

In Chapter 3, you learned that a *midsegment* of a triangle is a segment connecting the midpoints of two sides. In this lesson you will investigate properties of midsegments.

Investigation 1: Triangle Midsegment Properties

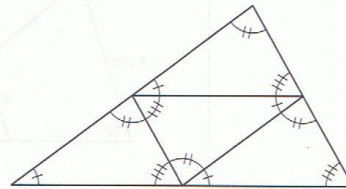
Follow Steps 1–3 in your book. Your conclusions should lead to the following conjecture.

Three Midsegments Conjecture The three midsegments of a triangle divide it into four congruent triangles.

C-41

Mark all the congruent angles in your triangle as shown in this example.

Focus on one of the midsegments and the third side of the triangle (the side the midsegment doesn't intersect). Look at the pairs of alternate interior angles and corresponding angles associated with these segments. What conclusion can you make? Look at the angles associated with each of the other midsegments and the corresponding third side.



Now compare the length of each midsegment to the length of the corresponding third side. How are the lengths related?

State your findings in the form of a conjecture.

Triangle Midsegment Conjecture A midsegment of a triangle is _____ to the third side and _____ the length of the third side.

C-42

The midsegment of a trapezoid is the segment connecting the midpoints of the two nonparallel sides.

Investigation 2: Trapezoid Midsegment Properties

Follow Steps 1–3 in your book. You should find that the trapezoid's base angles are congruent to the corresponding angles at the midsegment. What can you conclude about the relationship of the midsegment to the bases?



Now follow Steps 5–7. You should find that the midsegment fits twice onto the segment representing the sum of the two bases. That is, the length of the

(continued)

Lesson 5.4 • Properties of Midsegments (continued)

midsegment is half the sum of the lengths of the two bases. Another way to say this is: The length of the midsegment is the average of the lengths of the bases.

Use what you have learned about the midsegment of a trapezoid to complete this conjecture.

Trapezoid Midsegment Conjecture The midsegment of a trapezoid is _____ to the bases and equal in length to _____.

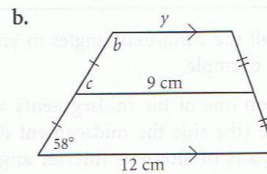
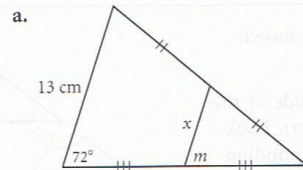
C-43

Read the text below the investigation on page 277 of your book and study the software construction. Make sure you understand the relationship between the Trapezoid and Triangle Midsegment Conjectures.

Work through the following example yourself before checking the solution.

EXAMPLE

Find the lettered measures.



► Solution

- a. By the Triangle Midsegment Conjecture, $x = \frac{1}{2}(13 \text{ cm}) = 6.5 \text{ cm}$.
The Triangle Midsegment Conjecture also tells you that the midsegment is parallel to the third side. Therefore, the corresponding angles are congruent, so $m = 72^\circ$.
- b. By the Trapezoid Midsegment Conjecture, $\frac{1}{2}(12 + y) = 9$. Solving for y gives $y = 6$.
The Trapezoid Midsegment Conjecture also tells you that the midsegment is parallel to the bases. Therefore, the corresponding angles are congruent, so $c = 58^\circ$.
By the Trapezoid Consecutive Angles Conjecture, $b + 58^\circ = 180^\circ$, so $b = 122^\circ$.

Properties of Parallelograms

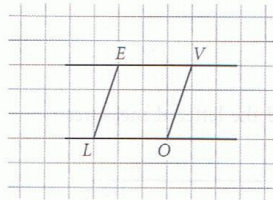
In this lesson you will

- Discover how the **angles** of a parallelogram are related
- Discover how the **sides** of a parallelogram are related
- Discover how the **diagonals** of a parallelogram are related

You have explored properties of kites and trapezoids and of the midsegments of triangles and trapezoids. In this lesson you will explore properties of parallelograms.

Investigation: Four Parallelogram Properties

Follow the directions in Step 1 in your book to construct and label a parallelogram.



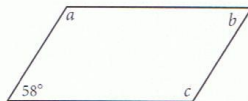
Use patty paper or a protractor to compare the measures of the opposite angles. Then use your findings to complete this conjecture.

Parallelogram Opposite Angles Conjecture The opposite angles of a parallelogram are _____ **C-44**

Consecutive angles are angles that share a common side. In parallelogram $LOVE$, $\angle LOV$ and $\angle EVO$ are one pair of consecutive angles. There are three other pairs. Find the sum of the measures of each pair of consecutive angles. You should find that the sum is the same for all four pairs. What is the sum? Complete this conjecture.

Parallelogram Consecutive Angles Conjecture The consecutive angles of a parallelogram are _____ **C-45**

Suppose you are given the measure of one angle of a parallelogram. Describe how you can use the conjectures above to find the measures of the other three angles. If you don't know, look at this particular figure. What are the values of a , b , and c ? (Remember all your parallel lines conjectures.)



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Lesson 5.5 • Properties of Parallelograms (continued)

Use a compass or patty paper to compare the lengths of the opposite sides of your parallelogram. How are the lengths related? Complete this conjecture.

Parallelogram Opposite Sides Conjecture The opposite sides of a parallelogram are _____.

C-46

Now draw the diagonals of your parallelogram. Label the point where the diagonals intersect M . How do LM and VM compare? How do EM and OM compare? What does this tell you about the relationship between the diagonals? Complete this conjecture.

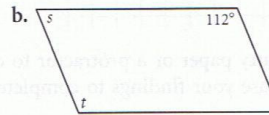
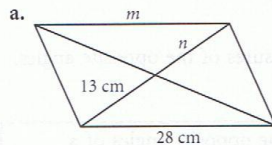
Parallelogram Diagonals Conjecture The diagonals of a parallelogram _____ each other.

C-47

In your book, read the text about **vectors** that follows the investigation.

Here is an example using your new conjectures.

EXAMPLE In parts a and b, the figures are parallelograms. Find the lettered measures and state which conjectures you used.



- **Solution**
- a. By the Parallelogram Opposite Sides Conjecture, $m = 28$ cm.
By the Parallelogram Diagonals Conjecture, $n = 13$ cm.
 - b. By the Parallelogram Opposite Angles Conjecture, $t = 112^\circ$.
By the Parallelogram Consecutive Angles Conjecture, $s = 180^\circ - 112^\circ = 68^\circ$.

Properties of Special Parallelograms

In this lesson you will

- Discover properties of **rhombuses** and their diagonals
- Discover properties of **rectangles** and their diagonals
- Discover properties of **squares** and their diagonals

In Lesson 5.5, you investigated parallelograms. In this lesson you focus on three special parallelograms—rhombuses, rectangles, and squares.

Investigation 1: What Can You Draw With the Double-Edged Straightedge?

Follow Steps 1–3 in your book. You should find that all the sides of the parallelogram you create are the same length. Use your findings to complete this conjecture.

Double-Edged Straightedge Conjecture If two parallel lines are intersected by a second pair of parallel lines that are the same distance apart as the first pair, then the parallelogram formed is a _____.

C-48

Now that you know a quick way to construct a rhombus, you will explore some special properties of rhombuses.

Investigation 2: Do Rhombus Diagonals Have Special Properties?

In this investigation you will look at the diagonals of a rhombus. Follow Steps 1 and 2 in your book. Then complete this conjecture.

Rhombus Diagonals Conjecture The diagonals of a rhombus are _____ and they _____ each other.

C-49

Follow Step 3 to compare the two angles formed at each vertex by a diagonal and the sides. Then complete this conjecture.

Rhombus Angles Conjecture The diagonals of a rhombus _____ the angles of the rhombus.

C-50

You have just explored rhombuses, parallelograms with four congruent sides. Now you will look at rectangles, parallelograms with four congruent angles.

By the Quadrilateral Sum Conjecture, you know that the sum of the angle measures of a rectangle is 360° . Because all the angles have the same measures, each angle must have measure 90° . In other words, a rectangle has four right angles.

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Lesson 5.6 • Properties of Special Parallelograms (continued)

Investigation 3: Do Rectangle Diagonals Have Special Properties?

Follow Steps 1 and 2 in your book. What do you notice about the lengths of the two diagonals? Because a rectangle is a parallelogram, you also know that the diagonals bisect each other. You can use your compass to confirm this for your rectangle. Combine these two observations to complete the conjecture.

Rectangle Diagonals Conjecture The diagonals of a rectangle are _____ and _____ each other. **C-51**

A square is a parallelogram that is both equiangular and equilateral. Here are two definitions of a square.

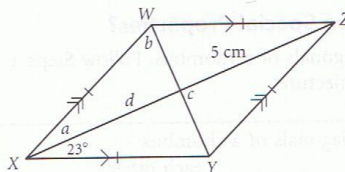
A **square** is an equiangular rhombus.

A **square** is an equilateral rectangle.

Because a square is a parallelogram, a rhombus, and a rectangle, all the properties of these quadrilaterals are also true for squares. Look back at what you know about the diagonals of each of these quadrilaterals, and use your findings to complete this conjecture.

Square Diagonals Conjecture The diagonals of a square are _____, _____, and _____. **C-52**

EXAMPLE Find the lettered measures.



► **Solution** The figure is a rhombus, so by the Rhombus Angles Conjecture, the diagonals bisect the angles. Therefore, $a = 23^\circ$.

By the Parallelogram Consecutive Angles Conjecture, $\angle WXY$ and $\angle XWZ$ are supplementary, so $m\angle XWZ + 46^\circ = 180^\circ$. Therefore, $m\angle XWZ = 134^\circ$. So, using the Rhombus Angles Conjecture, $b = \frac{1}{2}(134^\circ) = 67^\circ$.

By the Rhombus Diagonals Conjecture, the diagonals are perpendicular and bisect each other, so $c = 90^\circ$ and $d = 5$ cm.

5.7

Proving Quadrilateral Properties

In this lesson you will

- Learn about the “thinking backward” strategy for writing proofs
- Prove many of the quadrilateral conjectures from this chapter

In this chapter you have made many conjectures about the properties of quadrilaterals. In this lesson you will write proofs for several of these conjectures.

Look at the illustration of the firefighters in your book. The firefighter holding the hose has asked the other firefighter to turn on one of the hydrants. Which one should he turn on? One way to solve this problem is to start with the end of the hose in the first firefighter’s hand and trace it backward to a hydrant. You can use a similar strategy to help you write proofs.

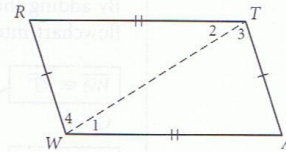
To plan a proof, it often helps to start with the conclusion (that is, the statement you want to prove) and work your way back to the beginning, one step at a time. Making a flowchart can help you visualize the flow of reasoning. The example in your book illustrates how this strategy works. Read this example carefully. Then read the example below, which shows how to prove the statement in Exercise 3.

EXAMPLE

Prove the conjecture: If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Given: Quadrilateral $WATR$ with $\overline{WA} \cong \overline{RT}$ and $\overline{WR} \cong \overline{AT}$, and diagonal \overline{WT}

Show: $WATR$ is a parallelogram

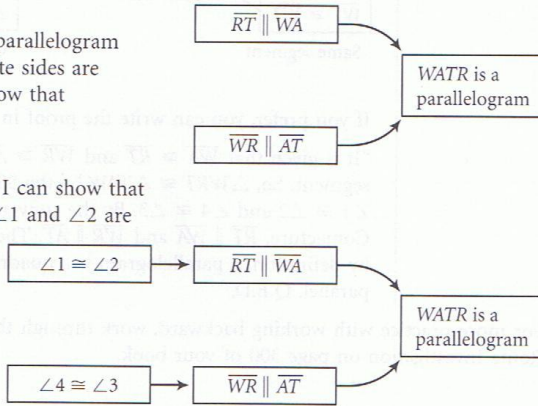


► **Solution**

Construct your proof by working backward. Your thinking might go something like this:

“I can show that $WATR$ is a parallelogram if I can show that the opposite sides are parallel. That is, I need to show that $\overline{RT} \parallel \overline{WA}$ and $\overline{WR} \parallel \overline{AT}$.”

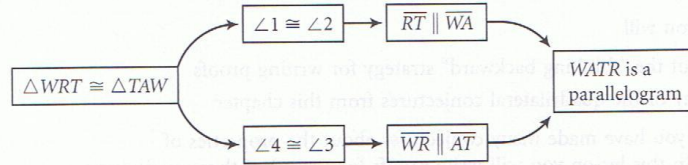
“I can show that $\overline{RT} \parallel \overline{WA}$ if I can show that the alternate interior angles $\angle 1$ and $\angle 2$ are congruent. I can show that $\overline{WR} \parallel \overline{AT}$ if I can show that the alternate interior angles $\angle 4$ and $\angle 3$ are congruent.”



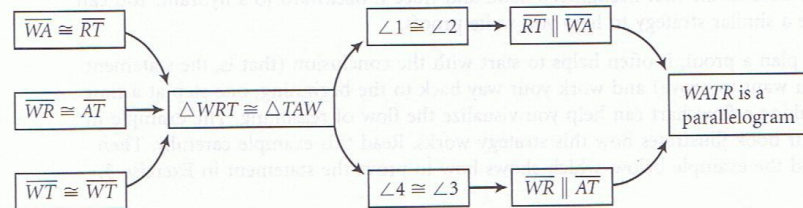
(continued)

Lesson 5.7 • Proving Quadrilateral Properties (continued)

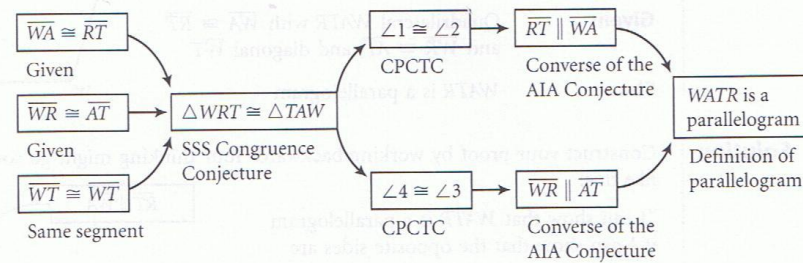
“I can show that $\angle 1 \cong \angle 2$ and $\angle 4 \cong \angle 3$ if they are corresponding parts of congruent triangles.”



“Can I show that $\triangle WRT \cong \triangle TAW$? Yes, I can, by SSS, because it is given that $\overline{WA} \cong \overline{RT}$ and $\overline{WR} \cong \overline{AT}$, and $\overline{WT} \cong \overline{WT}$ because it is the same segment in both triangles.”



By adding the reason for each statement below each box, you can make the flowchart into a complete flowchart proof.



If you prefer, you can write the proof in paragraph form:

“It is given that $\overline{WA} \cong \overline{RT}$ and $\overline{WR} \cong \overline{AT}$. $\overline{WT} \cong \overline{WT}$ because it is the same segment. So, $\triangle WRT \cong \triangle TAW$ by the SSS Congruence Conjecture. By CPCTC, $\angle 1 \cong \angle 2$ and $\angle 4 \cong \angle 3$. By the converse of the Alternate Interior Angles Conjecture, $\overline{RT} \parallel \overline{WA}$ and $\overline{WR} \parallel \overline{AT}$. Therefore, $WATR$ is a parallelogram because, by definition, a parallelogram is a quadrilateral in which the opposite sides are parallel. Q.E.D.”

For more practice with working backward, work through the Finding the Square Route Investigation on page 300 of your book.